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Calculation by the Method of Finite Differences of the Temperature Distribution in Layered Tissues

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Abstract—A numerical approach to obtain the temperature distribution in layered tissues with thermal source and cooling inside the tissues is presented in this paper. This approach can be applied to problems with a distributed internal thermal source produced by external radiation such as microwave, ultrasound, and shortwave, or by internal increase of metabolic rate in live subjects. The exact mechanism of cooling is not discussed here, but it is assumed to be due to blood flow *in vivo*. The calculation of the temperature distribution in layered tissues is based on a linear model of the tissues that consists of several layers of different kinds of tissues separated by parallel boundaries. Some simple mathematical forms of the cooling functions are also assumed in this paper. The results obtained agree well with six available experiments.

I. Introduction

AN EFFICIENT and safe application of diathermy depends on the knowledge of the relative heating pattern in layered tissues as they are subjected to treatment. The ultimate goal in diathermy is the production of a required temperature distribution in the tissues by an appropriate applicator.

The methods for studying temperature distribution so far have been mostly experimental. The tissues are exposed to an electromagnetic or ultrasound energy source for a period of time, and the temperature is measured at various points away from the applicator by thermal probes such as thermistors or thermocouples. The temperature distributions in layered tissue obtained in previous experimental studies (which are often referred to as the relative heating patterns by previous workers) accounted for only the thermal energy generated around the thermal probe [6], [7]. The thermal diffusion process taking place inside the tissues during the period of exposure to external energy source had been neglected in previous experiments.

Since the tissue parameters such as dielectric constants and loss tangents in the electromagnetic case and sound velocities and attenuations in the ultrasonic case are known [1]-[5], we can obtain the relative heating pattern by knowing the input strength of the energy source. By utilizing a linear mathematical model of the tissues and some simple assumptions on the boundary temperatures, the temperature distribution in various layers of tissue is obtained by solving the heat flow equation by the method of finite difference.

II. Mathematical Model and Formulation

The analytical model for this thermal diffusion problem consists of three plane layers of tissue with various thicknesses as shown in Fig. 1. Each layer contains: 1) a thermal source function $H_i(x)$ which is the heat power/unit volume converted from electromagnetic or ultrasonic power loss inside the tissues, and 2) a cooling function $C_i(x, T)$ which is the power/unit volume carried out by the blood circulation *in vivo*. The cooling function is related to the distribution of blood vessels in the tissues. The detailed mathematical form of the cooling function is discussed later.

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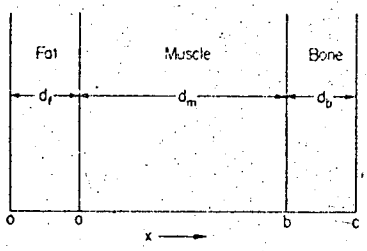

Fig. 1. Analytical model for thermal diffusion. $H_i(x)$ = thermal source function, $C_i(x, t)$ = cooling function, ρ_i = density of the tissue, S_i = specific heat, κ_i = thermal conductivity, k_i = diffusion constant ($i = f, b, m$ for fat, bone, and muscle, respectively).

The temperature distribution is obtained from the equation

$$\frac{\partial^2 T_i}{\partial x^2} - \frac{1}{\kappa_i}\frac{\partial T_i}{\partial t} = -\frac{1}{k_i}[H_i(x) - C_i(x, T)] \qquad (1)$$

where T_i is the temperature distribution in the layered tissues, i corresponds to fat, muscle, and bone (abbreviated as f, m, and b, respectively), k_i is the thermal conductivity, and κ_i is the diffusion constant. The boundary conditions for the temperature distribution functions T_i are as follows:

$$T_f(0) = \text{skin temperature} = T_s \qquad (2)$$

$$T_f(a) = T_m(a) \qquad (3)$$

$$T_m(b) = T_b(b) \qquad (4)$$

$$T_b(c) = \text{bone marrow temperature} = T_{bm} \qquad (5)$$

where a, b, and c are the coordinates of the interfaces as shown in Fig. 1. The temperature T_s of the skin is assumed to be constant throughout the treatment period. The temperature of the skin around the treatment area is maintained fairly constant either by cooling or by the heat capacity of the coupling medium. The temperature T_{bm} of the bone marrow is also assumed constant. There is a large amount of blood in the bone marrow that acts as a heat reservoir to keep the bone marrow temperature fairly constant. Moreover, most of the power generated from the external energy source has already been dissipated in the tissues before it reaches the bone marrow. These assumptions are justified by experiments performed previously [6]. The additional boundary condition required to specify the temperature distribution functions uniquely is that the heat flux must be continuous across the boundary. For the two boundaries, the conditions are as follows:

$$k_f \left.\frac{\partial T_f}{\partial x}\right|_{x=a} = k_m \left.\frac{\partial T_m}{\partial x}\right|_{x=a} \qquad (6)$$

$$k_m \left.\frac{\partial T_m}{\partial x}\right|_{x=b} = k_b \left.\frac{\partial T_b}{\partial x}\right|_{x=b}. \qquad (7)$$

The usual analytical approach to solve the inhomogeneous diffusion equation as in (1) is to employ the Green's function with appropriate boundary conditions (in the present case, the boundary condition will be of the Dirichlet type) [11]. The analytical solution requires the solution of three simultaneous integral equations and an approximation of the initial temperature distribution. Thus a direct numerical method of calculation was chosen.

Using the finite difference method, (1) is written as

$$T^i_{u,j+1} = \kappa^i \frac{\Delta t}{(\Delta x)^2}[T^i_{u-1,j} - 2T^i_{u,j} + T^i_{u+1,j}] + T^i_{u,j} + \Delta t \frac{\kappa_i}{k_i}(H^i_u - C^i_u) \qquad (8)$$

where the superscript i indicates the ith layer of tissue, and the subscripts u and j are the increment indices for the spatial coordinate x and the time coordinate t, respectively.

The criterion for convergence and stability of the solution of (8) is

$$\frac{\Delta t}{(\Delta x)^2} \leqslant \frac{1}{2}. \qquad (9)$$

The values of the thermal diffusion constant and thermal conductivity as measured by Lehmann [4] are used in this paper.

III. Thermal Source Function and Cooling Function

The thermal source function $H_i(x)$ in (1) describes the amount of heat power converted from other types of radiating power, such as electromagnetic, ultrasonic, etc. Different tissues absorb different modalities of power in a different manner. Thus it is expected that, in general, the relative amplitude of $H_i(x)$ will be different for each different i. Generally, the amount of power loss in tissue per unit volume when electromagnetic or ultrasonic waves propagate through the tissue varies approximately with distance from the energy source as

$$H_i(x) = A_i e^{-2w_i x}, \quad i = f, m, b \qquad (10)$$

when the amount of power reflected from the interface is small [1]. A_i is the amplitude of the exponential function and w_i is the attenuation constant of the tissue in the ith medium. These constants vary with the type of external energy sources. (See Table I.)

If all power energy losses in tissues are converted into heat, the integration of (10) with respect to x over the entire fat tissue, as shown in Fig. 1, is the total thermal power generated in the fat tissue. By a similar procedure, the total thermal power generated in muscle and bone can be obtained. Since the A_i are related to one another through the relative heating pattern, their values can be calculated by setting the total thermal power generated in the tissues equal to the total input power. Some typical relative heating patterns for this problem are shown in Fig. 2.

Present knowledge of cooling mechanisms *in vivo* is insufficient to determine the cooling function. However, if the hypothesis that the cooling is largely due to blood circulation is accepted, it would be reasonable that the cooling function should be a function of space, since it is well known that the distribution of vessels in tissues is a function of the space.

Throughout the following calculations, the density distribution of vessels is assumed uniform within the same layer of tis-

TABLE I
ACOUSTICAL, ELECTROMAGNETIC, AND THERMAL PROPERTIES OF TISSUES

Type of Tissue	Acoustical Attenuation (1 MHz) nepers/cm	Acoustical Velocity of Propagation (1 MHz) meters/sec	Electromagnetic Wave Attenuation (3 GHz) nepers/cm	Electromagnetic Propagation Constant (3 GHz) radians/cm	Density gm/cm^3	Specific Heat cal/gm-°C	Thermal Conductivity cal-cm/ cm^2-sec-°C
Fat	0.04	1,476	0.11	1.3	0.97	0.621	0.00055
Muscle	0.12	1,568	0.56	4.6	1.07	0.750	0.0012
Bone	0.96	3,360	-	-	1.70	0.504	0.0035

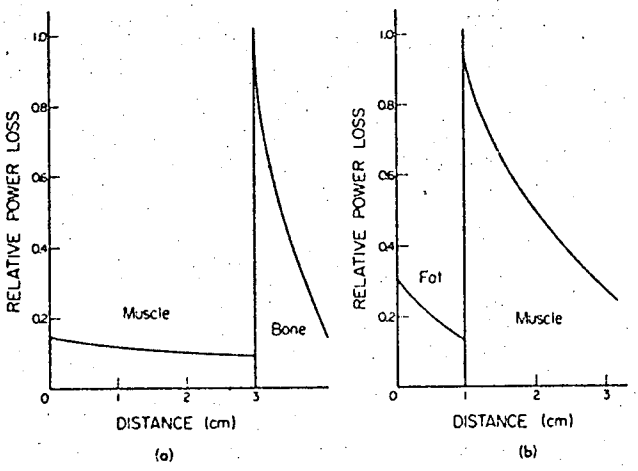

Fig. 2. Relative power loss by (a) ultrasound and (b) microwave. The relative power loss is the power per unit volume converted into heat and normalized to the maximum value.

sue of a specimen. Theoretical results thus obtained are compared with those of previous experiments [6], [13].

The dependence on temperature of the cooling function is assumed to be power series in $(T - T_o)$ where T_o is the initial temperature. Only the first-order and second-order terms are included in the cooling function. It is also postulated that when a threshold temperature T_c is reached a different mechanism of cooling is triggered. The cooling function for this condition is discussed later. The cooling function can then be written as

$$C_i(x,T) = S_i(T - T_o) + [\beta_i(T - T_o)^2]\, f(t), \quad i = f,m,b \quad (11)$$

where S_i and β_i are constants to be determined, and $f(t)$ is an appropriate function of time.

IV. Computational Results

The experimental results obtained by Lehmann with pigs irradiated by ultrasound are used to compare with calculations using (8). A computer program[1] was developed to implement the numerical calculations.

[1]This computer program is available upon request.

Two cases were studied: first, the specimen is sacrificed, thus the blood circulation is absent; second, the specimen is anesthetized and circulation is present.

In the first case, the specimen is sacrificed with an overdose of Numbutar. The initial temperature distribution is approximated by straight-line interpolation between points of known temperature and programmed into the computer as $T_{u,o}$.

The incident wave in all calculations is normal to the interface. Therefore only longitudinal waves are considered.

The calculated values of A_i in (10) are

$$A_b = 85.0 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3$$
$$A_m = 12.7 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3$$
$$A_f = 4.0 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3.$$

The intensity of the incident wave was assumed to be 0.225 W/cc. A better fit of the calculated values is obtained if instead of the calculated value of A_m, the following is used: $A_m = 8.0 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3$.

Fig. 3 shows the calculated and experimental results, after the specimen has been irradiated for 5 min. The agreement between calculated and experimental results is within 3°C standard deviation.

The calculated and measured maximum temperatures seem to occur at different locations in the bone. This difference may stem from the difficulty of inserting thermocouples in the cortical bone, and thus missing the experimental maximum.

In the second case, every condition was maintained as in the first except that the specimen was alive, and consequently, circulation of blood was present.

By trial and error the best fit of the coefficients of (11) was obtained when $\beta_i = 0$ and

$$S_m = 2.0 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3 \cdot °\text{C}$$
$$S_b = 2.5 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3 \cdot °\text{C}$$

since there is little blood circulation if fat, S_f was assumed to be zero.

Fig. 4 depicts curve C, which best fits the experimental data, as well other curves which show the effect of other choices of coefficients. A change in S_m affects the temperature distribu-

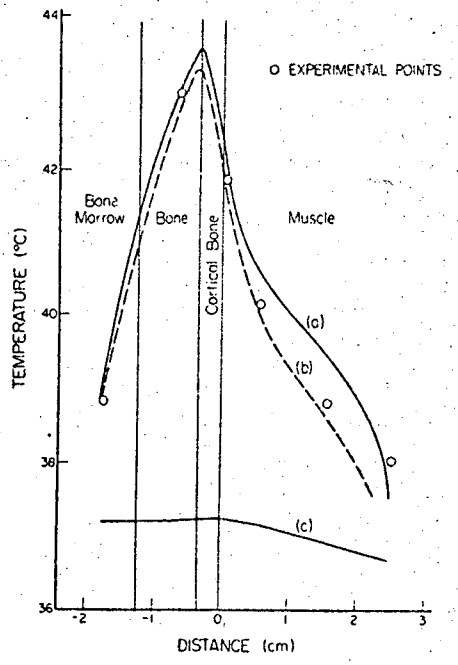

Fig. 3. Calculated temperature distribution in a pig thigh (dead specimen) after 5 min of ultrasound irradiation. (a) $A_m = 12.7$ kg · cal/s · m^3. (b) $A_m = 8.00$ kg · cal/s · m^3. (c) Initial temperature distribution.

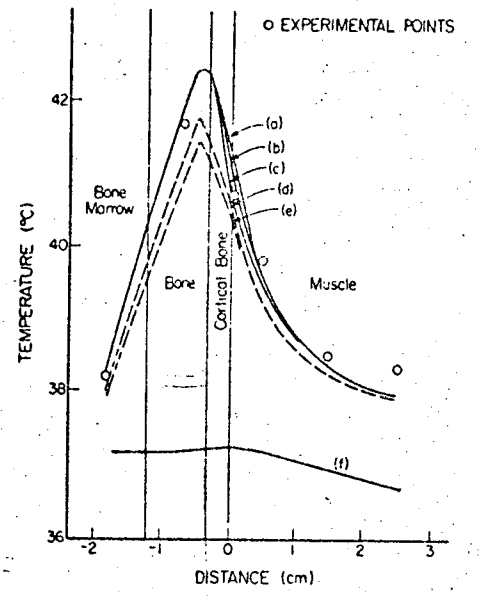

Fig. 4. Calculated temperature distribution in a pig thigh (live specimen) after 5 min of ultrasound irradiation. (a) $S_m = 0$, $S_b = 2.5$ kg · cal/s · m^3 · °C. (b) $S_m = 2.0$ kg · cal/s · m^3 · °C, $S_b = 2.5$ kg · cal/s · m^3 · °C. (c) $S_m = 8.0$ kg · cal/s · m^3 · °C, $S_b = 2.5$ kg · cal/s · m^3 · °C. (d) $S_m = 2.0$ kg · cal/s · m^3 · °C, $S_b = 4.0$ kg · cal/s · m^3 · °C. (e) $S_m = 2.0$ kg · cal/s · m^3 · °C, $S_b = 5.0$ kg · cal/s · m^3 · °C. (f) Initial temperature distribution.

tion only in the vicinity of the bone-muscle interface, whereas a change S_b causes an overall change in temperature distribution.

The temperature change in two locations as a function of time is shown in Figs. 5 and 6. The effect of circulation is also shown in these figures. In Fig. 5 the results of two experiments on the dead specimen and two experiments on the live specimen are shown. Obviously, in the dead specimen the circulation is absent; thus there is no cooling effect.

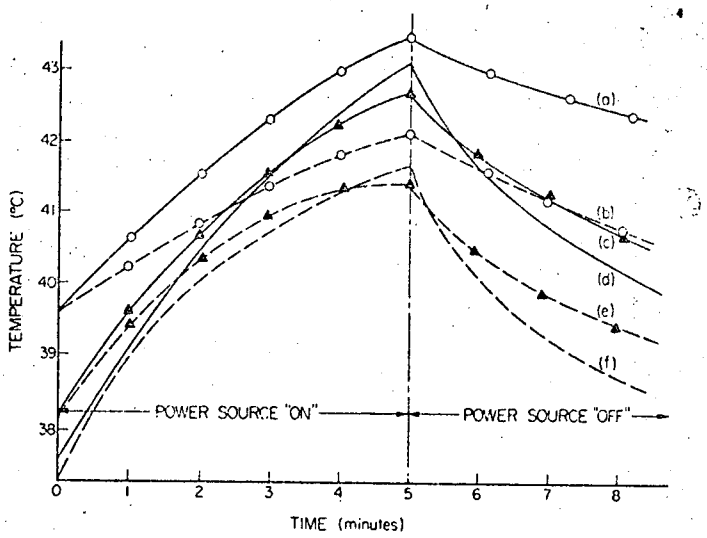

Fig. 5. Calculated and experimental results of the temperature rise as a function of time. The location of the thermocouple is −0.75 cm from the bone surface. (a) Experimental result 1 without cooling. (b) Experimental result 1 with cooling. (c) Experimental result 2 without cooling. (d) Calculated result without cooling. (e) Experimental result 2 with cooling. (f) Calculated result with cooling.

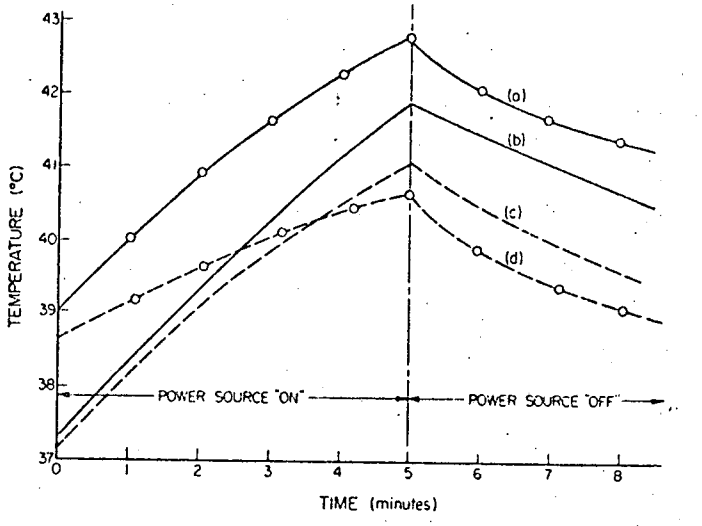

Fig. 6. Calculated and experimental results of the temperature rise as a function of time. The location of the thermocouple is at the bone surface. (a) Experimental temperature without cooling. (b) Calculated temperature without cooling. (c) Experimental temperature with cooling. (d) Calculated temperature with cooling.

The last calculation made in this study was based on an experiment in which human thighs were exposed to 915-MHz microwave radiation. In this experiment, the temperature at various distances from the microwave energy source was measured at 2-min intervals after the source was energized. The average power density of the microwave energy source is approximately equal to 0.3 W/cm^2. Based on the assumption that all microwave energy is absorbed by the tissue, the amplitudes of the thermal source functions are calculated to be

$$A_m = 40 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3$$
$$A_f = 12 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3$$
$$A_b = 0.$$

Previous experimental results have shown that the muscle temperature rose to 43-45°C and then decayed to a steady-state value of 42°C with the external energy source kept on. These observed phenomena indicate that there must be a change of the cooling function *in vivo* when the temperatures of the tissues reach a certain critical point. Experimental results further indicate that the temperature of the tissues does not drop off suddenly due to the change of cooling function, but rises and then drops exponentially to a steady-state temperature. The continued rise of the temperature in muscle tissues shows that there may be a time delay before the change of cooling is fully effective. Based on these observed phenomena, the cooling function is constructed.

The cooling function in fat tissue is assumed to be zero since the density of blood vessels in fat tissue is small. The cooling function in muscle tissue takes the same form as in (11); the second-order term is equal to zero until the temperature of the tissue reaches a critical point. Mathematically, the cooling function in muscle is expressed as

$$C_m(x, T) = S_m(T - T_o), \quad T < T_c$$

$$C_m(x, t) = S_m(T - T_o) + [\beta_m(T - T_o)^2] \cdot \left[1 - \exp\left(\frac{-(t - t_o)}{t_c}\right)\right], \quad T > T_c \quad (12)$$

where S_m and β_m are constants to be determined, t_o is the time when the tissue temperature reaches its critical value T_c, and t_c is a time constant. By trial and error, it was found that if $S_m = 1.1$ kg · cal/s · m^3 · °C,

$$\beta_m = 0.7 \text{ kg} \cdot \text{cal/s} \cdot \text{m}^3 \cdot {}^\circ\text{C}$$
$$T_c = 42^\circ\text{C}$$
$$t_c = 240 \text{ s}$$

the calculation would match the experimental results well. A comparison of the calculated and experimental results is shown in Fig. 7.

V. Conclusion

Temperature distribution inside the layered tissues radiated by external energy sources such as microwave and ultrasound is obtained by solving the thermal diffusion equation by the method of finite difference. A linear mathematical model is employed in this calculation for the tissues. The thermal source functions corresponding to the heat energy converted from the other form of energy in various tissues are assumed to be exponential functions with unequal amplitudes and decay constants. The cooling functions in various tissues *in vivo* are assumed to be linear functions of $(T - T_o)$. A second-order term of $(T - T_o)$ is included in the cooling function only when the tissue temperature reaches a critical temperature. A time delay factor is used with the extra cooling effects. This factor may be explained by the fact that it requires a certain time for the physiological reaction to take place when it is first triggered by the temperature of the tissue reaching a critical temperature. The results obtained from calculation agree with the experimental observations by other authors within the standard deviation of the experiment in both cases where the cooling function is present or absent.

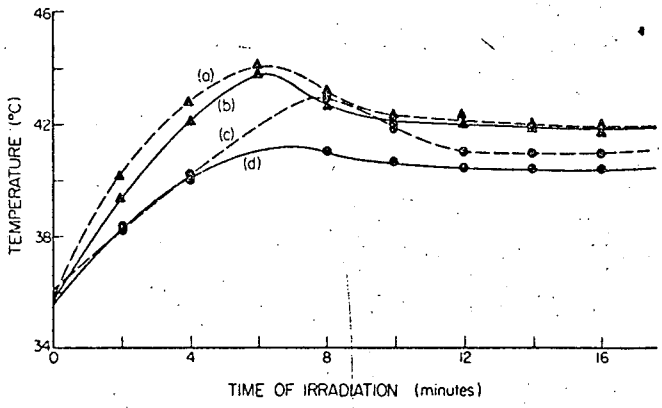

Fig. 7. Temperature rise in human thigh as a function of time (915-MHz microwave radiation). (a) and (c) Experimental results. (b) and (d) Calculated results.

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